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Stochastic calculus in superspace: I. Supersymmetric Hamiltonians*

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Abstract. Various analytic results which combine fermionic Brownian motion with stochastic integration are described, and it is shown that a wide class of stochastic diffferential equations in superspace have solutions. Such solutions are then used to derive a Feynman–Kac formula for a supersymmetric system in terms of the supercharge whose square is the Hamiltonian of the system. This is achieved by introducing superpaths parametrized by a commuting and an anticommuting time variable.

1. Introduction

Superspaces are spaces parametrized by commuting and anticommuting variables. By considering paths in superspace, the standard Feynman method for path integral quantization may be extended to systems with fermionic degrees of freedom. The purpose of this paper is to investigate the stochastic calculus of these generalized Brownian paths, and hence to extend the range of Hamiltonians which can be studied by these methods. Additionally, where the theories are supersymmetric, the path integrals will be presented in a manifestly supersymmetric formalism.

Classical stochastic calculus is based on a definition of integration along Brownian paths b_t. The integrals obtained have many applications, leading to the theory of stochastic differential equations and an analytic treatment of white noise, as well as giving new insights into various second-order differential operators, and into Hamiltonian systems in quantum mechanics. They also allow the extension of path integration techniques from Euclidean space to more general Riemannian manifolds.

Recently, following the ideas of Martin [1], this author has considered a fermionic analogue of Brownian paths involving paths in anticommuting space [2, 3], and also a theory of superpath integration [4]. (Superpaths $b_t + \tau\xi_t$ are paths parametrized both by ordinary time t and by anticommuting time τ; the theory of superpath integration is appropriate for considering supersymmetric systems, where the Hamiltonian operator H is the square of a supercharge operator Q. Superpaths are also considered by Haba [5] and by Friedan and Windey [6].)

There are two essential steps in the fermionic path integration introduced in [2]; the first is to observe that the canonical anticommutation relations satisfied by fermionic operators in quantum mechanics can be represented by differential operators on spaces

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of functions of anticommuting variables. This representation is the analogue of the Schrödinger representation of the canonical commutation relations for bosonic quantum mechanics. The second step is to construct a measure (in a generalized sense) on the space of paths in anticommuting space which leads to a Feynman–Kac formula for a useful class of Hamiltonians. A detailed description of these ideas may be found in [7]. In this paper the standard theory of stochastic calculus is extended to include fermionic paths and superpaths, and consequently to extend the class of fermionic and supersymmetric quantum-mechanical systems which can be studied using these path integrals.

Despite its many uses, stochastic calculus is often unfamiliar to theoretical physicists. This is possibly because much of the literature is only accessible to those who are well-versed in mathematical probability theory (although there are notable exceptions, such as the books by Simon [8], Arnold [9] and Øksendal [10]). This paper does not assume any prior knowledge of stochastic calculus.

In their simplest form, path integral methods can be applied to Hamiltonians on $\mathbb{R}^m$ of the form

$$H = -\tfrac{1}{2}\sum_{i=1}^{m} \partial_i \partial_i + V(x) \tag{1.1}$$

leading to the Feynman–Kac formula

$$\exp(-Ht)f(x) = \int \mathrm{d}\mu \, \exp\left(-\int_0^t V(x+b_s)\mathrm{d}s\right) f(x+b_t). \tag{1.2}$$

Here the measure $\mathrm{d}\mu$ is Wiener measure, often written in the physics literature as

$$\mathcal{D}x \exp\left(-\int_0^t \tfrac{1}{2}\dot{x}^2 \,\mathrm{d}t\right). \tag{1.3}$$

Stochastic calculus allows one to construct a Feynman–Kac formula for a Hamiltonian which is an arbitrary second-order elliptic differential operator. In fact there are two methods by which stochastic calculus can extend the range of possible Hamiltonians. First, by including a term of the form $\exp(-\int_0^t \sum_{a=1}^m f_a(s)\,\mathrm{d}b_s^a)$ in the integrand, a Feynman–Kac formula for Hamiltonians containing first-order derivatives may be obtained. Second, if the simple Brownian paths b_t are replaced by paths x_t which satisfy the stochastic differential equation

$$\mathrm{d}x^j = f^j(t)\mathrm{d}t + \sum_{a=1}^{m} c^j{}_a(t)\mathrm{d}b_t^a \tag{1.4}$$

then a Feynman–Kac formula may be obtained for Hamiltonians which are arbitrary second-order elliptic operators, the second-order part being

$$-\tfrac{1}{2}\sum_{a,j,k=1}^{m} c_a^j(x)c_a^k(x)\,\frac{\partial^2}{\partial x^j\,\partial x^k}. \tag{1.5}$$

The main result of this paper is a supersymmetric Feynman–Kac formula for Hamiltonians H which are the square of a supercharge Q, Q being a Dirac-like operator. The key idea is the use of superpaths, which allow one to work in terms of the supercharge directly, rather than in terms of the Hamiltonian. As in the classical case this Feynman–Kac formula is achieved by applying the Itô formula to carefully chosen functions of solutions to certain stochastic differential equations. Before establishing the main theorem, a number of technical steps must be taken.

Section 2 of this paper contains a brief review of fermionic Brownian motion and path integration, while section 3 considers stochastic integrals in the presence of fermionic paths; this section contains a proof of a formula for change of variable of integration which generalizes the usual Itô formula. Section 4 describes some results in calculus on a $(1,1)$-dimensional superspace, culminating in two supersymmetric Itô formulae for superpaths, while section 5 contains a crucial theorem on the existence of unique solutions to a useful class of stochastic differential equations in superspace. Finally, in section 6, the supersymmetric Feynman–Kac formula is established.

Because the canonical anticommutation relations are also the defining relations of a Clifford algebra, the methods developed in this paper are also applicable to various geometrical operators on spin bundles and bundles of differential forms on manifolds. These applications are the subject of a companion paper [11], where it is shown that the results of this paper lead to a natural theory of path integration on supermanifolds and spin bundles. This allows a rigorous derivation of the path integration results needed in the supersymmetric proofs of the index theorem given by Alvarez-Gaumé [12] and by Friedan and Windey [6].

The work described in this paper is less formal than the generalizations of Brownian motion considered in the various non-commuting probability theories [13]. The constructions are deliberately designed to provide a rigorous version of the powerful but non-rigorous fermionic path integrals used in fermionic and supersymmetric quantum mechanics. (Good accounts of these methods may be found in many places, for instance in [14] or [15].) Most closely related to this paper is the work of Haba [5], who develops a particular example of non-commutative probability which is formally equivalent to the superpaths of this paper in Euclidean space. A quite distinct approach to fermionic path integration is described by Gaveau and Schulman [16].

2. Grassmann preliminaries

This section introduces notation, and summarizes the important aspects of the approach to integration in infinite dimensional spaces of anticommuting variables introduced in [2]. In particular, fermionic Brownian motion is defined. Throughout this paper lower case roman letters will be used for quantities with even Grassmann parity, lower case greek letters for odd quantities, while roman capitals will be used for quantities which may have either parity.

For each positive integer L, $\mathbb{B}_L$ will denote the real Grassmann algebra on $\mathbb{R}^L$. Thus $\mathbb{B}_L$ is the algebra over the reals with generators $\mathbf{1}, \beta_1, \ldots, \beta_L$, and relations

$$\begin{aligned} \beta_i\beta_j &= -\beta_j\beta_i \qquad i,j = 1,\ldots,L \\ \beta_i\mathbf{1} &= \mathbf{1}\beta_i \qquad i = 1 = \beta_i,\ldots,L. \end{aligned} \tag{2.1}$$

The Grassmann algebra $\mathbb{B}_L$ has a Z_2 grading $\mathbb{B}_L = \mathbb{B}_{L,0} + \mathbb{B}_{L,1}$ where elements of $\mathbb{B}_{L,0}$ (respectively $\mathbb{B}_{L,1}$) are the sum of terms containing the product of an even (respectively odd) number of the anticommuting generators $\beta_1, \ldots, \beta_L$. Elements of $\mathbb{B}_{L,0}$ are said to be even while elements of $\mathbb{B}_{L,1}$ are said to be odd. Any two odd elements anticommute, while even elements commute both with one another and with odd elements. There is a canonical algebra homomorphism ϵ of $\mathbb{B}_L$ onto $\mathbb{R}$ which maps the unit element $\mathbf{1}$ to the real number 1 and maps the anticommuting generators $\beta_1,\ldots,\beta_L$ to zero. If m and n are positive integers $\mathbb{R}_L^{m,n}$ will denote $\mathbb{R}^m \times (\mathbb{B}_{L,1})^n$.

If J is any set, $\mathbb{R}_L^{(m,n)[J]}$ will denote $\prod_{j\in J} {}_j\mathbb{R}_L^{m,n}$ where ${}_j\mathbb{R}^{m,n} = \mathbb{R}^m \times ({}_j\mathbb{B}_{L,1})^n$, with ${}_j\mathbb{B}_L$ being the Grassmann algebra with L anticommuting generators ${}_j\beta_1,\ldots,{}_j\beta_L$ in addition to the commuting generator $\mathbf{1}$. The reason that $\mathbb{R}_L^{(m,n)[J]}$ will be used rather than the simpler $(\mathbb{R}_L^{m,n})^J$ is that many of the useful functions which one might wish to construct on the latter space will be zero for the quite trivial reason that repeated factors of Grassmann generators are zero. It is thus necessary to use different anticommuting generators for each $j \in J$. All the odd generators of this plethora of Grassmann algebras are defined to be mutually anticommuting, so that an odd element from one algebra will anticommute with an odd element from any other.

A typical element of $\mathbb{R}_L^{m,n}$ will be denoted $(x^1,\ldots,x^m,\theta^1,\ldots,\theta^n)$. For reasons which will become clear later, usually we will consider $\mathbb{R}_L^{(m,2n)[J]}$, and a typical element of $\mathbb{R}_L^{(m,2n)[J]}$ (where J is a finite set containing N elements) will be denoted $(\underline{x}^1,\ldots,\underline{x}^N,\underline{\theta}^1,\ldots,\underline{\theta}^N,\underline{\rho}^1,\ldots,\underline{\rho}^N)$ or $(x^{1,1},\ldots,x^{N,m},\theta^{1,1},\ldots,\theta^{N,n},\rho^{1,1},\ldots,\rho^{N,n})$.

Given any class of functions on $\mathbb{R}^m$, such as $C^\infty(\mathbb{R}^m,\mathbb{R})$ or $L^2(\mathbb{R}^m,\mathbb{C})$, one may readily construct an analogous class of functions on $\mathbb{R}_L^{m,n}$ in a standard manner. The corresponding classes of functions on $\mathbb{R}_L^{m,n}$ are denoted $C^{\infty\prime}(\mathbb{R}_L^{m,n},\mathbb{R})$, $L^{2\prime}(\mathbb{R}_L^{m,n},\mathbb{C})$ and so on. For instance, if $\mathbb{E}$ is a Banach space, $C^{\infty\prime}(\mathbb{R}_L^{m,n},\mathbb{E})$ is the class of functions

$$f : \mathbb{R}_L^{m,n} \to B_L \otimes \mathbb{E}$$

such that

$$f(x^1,\ldots,x^m,\theta^1,\ldots,\theta^n) = \sum_{\mu\in M_n} f_\mu(x^1,\ldots,x^m)\theta^{\mu_1}\ldots\theta^{\mu_k} \tag{2.2}$$

for some $f_\mu \in C^\infty(\mathbb{R}^m,\mathbb{E})$. (Here $\mu = \mu_1\ldots\mu_k$ is a multi-index with $1 \le \mu_1 < \ldots < \mu_k \le n$ and M_n is the set of all such multi-indices, including the empty one.) Where only sequences μ of even length contribute, f is said to be even. A function on $\mathbb{R}_L^{m,n}$ of this general form is said to be bounded if each of the coefficient functions f_μ are bounded. Differentiation of a function of $\mathbb{R}_L^{m,n}$ with respect to the ith even argument will be denoted ∂_i and differentiation with respect to the jth odd element ∂_{m+j} or δ_j.

The domain of spaces $C^{p\prime}(\mathbb{R}^{m,n},\mathbb{E})$ (where $p \ge 5$) can be extended to include even elements of any Grassmann algebra $\mathbb{B}_M$ (where M is a positive integer) by using Taylor expansions truncated to the first three terms. That is, if ϵ denotes the canonical mapping of $\mathbb{B}_M$ onto $\mathbb{R}$ and of $\mathbb{B}_M^m$ onto $\mathbb{R}^m$, then the extension of a function $f \in C^{p\prime}(\mathbb{R}^{m,n},\mathbb{E})$ with

$$f(x^1,\ldots,x^m,\theta^1,\ldots,\theta^n) = \sum_{\mu\in M_n} f_\mu(x^1,\ldots,x^m)\theta^{\mu_1}\ldots\theta^{\mu_k} \tag{2.3}$$

for some $f_\mu \in C^\infty(\mathbb{R}^m,\mathbb{E})$ is the function

$$f : (\mathbb{B}_{M,0})^m \times {\mathbb{B}_{L,1}}^n \to \mathbb{B}_L \otimes \mathbb{B}_{M,0} \otimes \mathbb{E}$$

with

$$f(x^1,\ldots,x^m,\theta^1,\ldots,\theta^n) = \sum_{\mu\in M_n} \tilde{f}_\mu(x^1,\ldots,x^m)\theta^{\mu_1}\ldots\theta^{\mu_k}$$

where

$$\tilde{f}_\mu(x) = f_\mu(\epsilon(x)) + \sum_{i=1}^{m}(x^i - \epsilon(x^i))\partial_i f_\mu(\epsilon(x))$$
$$+ \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(x^i - \epsilon(x^i))(x^j - \epsilon(x^j))\partial_i\partial_j f_\mu(\epsilon(x)). \tag{2.4}$$

Differentiability up to at least the fifth order is required so that these extended functions obey a Taylor theorem with a third-order remainder term. Further details of the analysis of functions of Grassmann variables may be found in [17].

Integration of functions of r anticommuting variables $\theta^1, \ldots, \theta^r$ is carried out according to the method of Berezin [18] where

$$\int \mathrm{d}^r\theta f(\theta) = f_{1\ldots r} \tag{2.5}$$

if $f(\theta) = f_{1\ldots r}\theta^1 \ldots \theta^r +$ terms of lower order in θ.

Combining this with Lebesgue–Stieltjes integration over the real variables enables one to integrate functions on $\mathbb{R}_L^{(m,n)[J]}$ when J is finite. An extension to integration when J is infinite has been defined in [2], using an analogue of the Kolmogorov construction [8]. In particular, if I is the interval $[0,t]$, n-dimensional Grassmann Wiener measure is defined on $\mathbb{R}_L^{(0,2n)[I]}$ by finite-dimensional distributions in the following way: suppose that J is a finite subset of I with $J = \{t_1, \ldots, t_N\}$ where $0 < t_1 < \ldots < t_N \le t$. Then the corresponding finite distribution is

$$f_J : \mathbb{R}_L^{(0,2n)[J]} \to \mathbb{B}_L \otimes {}_{t_1}\mathbb{B}_L \otimes \ldots \otimes {}_{t_N}\mathbb{B}_L$$
$$(\underline{\theta}^1, \ldots, \underline{\theta}^N, \underline{\rho}^1, \ldots, \underline{\rho}^N) \mapsto \exp[-\mathrm{i}(\underline{\rho}^1 \cdot \underline{\theta}^1 + \underline{\rho}^2 \cdot (\underline{\theta}^2 - \underline{\theta}^1) + \ldots$$
$$+ \underline{\rho}^N \cdot (\underline{\theta}^N - \underline{\theta}^{N-1}))]. \tag{2.6}$$

Here n is assumed to be an even number and $\underline{\rho}^r \cdot \underline{\theta}^r = \sum_{i=1}^{n} \theta^{r,i}\rho^{r,i}$. Expectations are calculated using Berezin integration. As always with Berezin integration there is no concept of a measurable set, only a formal action on functions. The space $\mathbb{R}_L^{(0,2n)[I]}$ equipped with this Grassmann measure will be called Grassmann Wiener space.

The fact that $\mathbb{R}_L^{(0,2n)[I]}$ is not simply a product of copies of $\mathbb{R}_L^{(0,2n)}$ makes a somewhat complicated definition of Grassmann random variables necessary [2].

Definition 2.1. For each $M = 1, 2, \ldots$, let J_M be a finite subset of I, with $J_M \subset J_{M+1}$ for each M. Also suppose that $G_M \in L^{2\prime}(\mathbb{R}_L^{(0,2n)[J_M]}, \mathbb{C})$, and let

$$I_M(G) = \int \mathrm{d}^{\sharp(J_M)}\theta\, \mathrm{d}^{\sharp(J_M)}\rho\, f_J(\underline{\theta}^1, \ldots, \underline{\theta}^{\sharp(J_M)}, \underline{\rho}^1, \ldots, \underline{\rho}^{\sharp(J_M)})$$
$$\times G_M(\underline{\theta}^1, \ldots, \underline{\theta}^{\sharp(J_M)}, \underline{\rho}^1, \ldots, \underline{\rho}^{\sharp(J_M)}). \tag{2.7}$$

Then the collection $(J_M, G_M : M = 1, 2, \ldots)$ is said to define a *Grassmann random variable* on Grassmann Wiener space if the sequence $(I_M(G))$ tends to a limit as M tends to infinity. This limit is denoted $\int \mathrm{d}\mu_f\, G$ or $\mathbb{E}_F(G)$. (A sequence of pairs $(J_M, G_M : M = 1, 2, \ldots)$ which does not necessarily satisfy the convergence condition is called a generalized Grassmann random variable.)

Suppose that $J = \{t_1, \ldots, t_N\}$ is a finite subset of I. Then any function $f \in L^{2\prime}(\mathbb{R}_L^{(0,2n)[J]})$ defines a Grassmann random variable with the sequence of pairs of sets and functions (J_M, G_M) being the constant sequence (J, f). Such Grassmann random variables are said to be finitely defined. For each s in I there are $2n$ finitely defined Grassmann random variables $\theta_s^i, \rho_s^i, i = 1, \ldots, n$, corresponding to $J = \{s\}$ and

$$\theta_s^i(\theta^1, \ldots \theta^n, \rho^1, \ldots \rho^n) = \theta^i \qquad \rho_s^i(\theta^1, \ldots \theta^n, \rho^1, \ldots \rho^n) = \rho^i. \tag{2.8}$$

The $(0, 2n)$-dimensional process $(\theta_s^1, \ldots \theta_s^n, \rho_s^1, \ldots \rho_s^n)$ is called fermionic Brownian motion.

The particular choice of distribution (2.2) is chosen because it relates directly to the action of differential operators on $L^{1\prime}(\mathbb{R}_L^{(0,n)})$. The connection is made by first observing that, if $f \in L^{1\prime}(\mathbb{R}_L^{(0,n)})$,

$$f(\phi) = \int \mathrm{d}^n\theta \prod_{i=1}^{n}(\theta - \phi) f(\theta) \tag{2.9}$$

(where $\phi^1, \ldots, \phi^n$ are anticommuting variables) and secondly that the delta function $\prod_{i=1}^{n}(\theta - \phi)$ is the Fourier transform of 1, that is

$$\prod_{i=1}^{n}(\theta - \phi) = \int \mathrm{d}^n\rho \exp\left(-\mathrm{i}\sum_{i=1}^{n}\rho^i(\theta^i - \phi^i)\right) \tag{2.10}$$

(where $\rho^1, \ldots, \rho^n$ are also anticommuting variables). Thus

$$f(\phi) = \int \mathrm{d}^n\rho\, \mathrm{d}^n\theta \exp\left(-\mathrm{i}\sum_{i=1}^{n}\rho^i(\theta^i - \phi^i)\right) f(\theta) \tag{2.11}$$

and so

$$\phi^i f(\phi) = \int \mathrm{d}^n\rho\, \mathrm{d}^n\theta\, \theta^i \exp\left(-\mathrm{i}\sum_{i=1}^{n}\rho^i(\theta^i - \phi^i)\right) f(\theta)$$

and

$$\delta_i f(\phi) = \int \mathrm{d}^n\rho\, \mathrm{d}^n\theta\, (-\mathrm{i})\rho^i \exp\left(-\mathrm{i}\sum_{i=1}^{n}\rho^i(\theta^i - \phi^i)\right) f(\theta) \tag{2.12}$$

recalling that δ_i denotes differentiation with respect to the ith anticommuting variable. Comparison with (2.3), together with the translational invariance of Berezin integration, then shows that, for s in I,

$$f(\phi) = \int \mathrm{d}\mu_f\, f(\theta_s + \phi) \qquad \phi^i f(\phi) = \int \mathrm{d}\mu_f\, (\theta_s^i + \phi) f(\theta_s + \phi) \tag{2.13}$$

$$\delta_i f(\phi) = \int \mathrm{d}\mu_f\, (-\mathrm{i})\rho_s^i f(\theta_s + \phi).$$

The canonical anticommutation relations of n fermionic operators $\psi^i, i = 1, \ldots, n$ are

$$\psi^i \psi^j + \psi^j \psi^i = 2\delta^{ij}. \tag{2.14}$$

These relations can be represented on the space $L^{1\prime}(\mathbb{R}_L^{(0,n)})$ of functions of n anticommuting variables $\theta^1, \ldots \theta^n$ by the operators $\psi^i = \theta^i + \partial/\partial\theta^i$. Comparison with (2.13) shows that

$$\psi^i f(\phi) = \int \mathrm{d}\mu_f\, \xi_s^i f(\theta_s + \phi) \tag{2.15}$$

where

$$\xi_s^i = \theta_s^i - i\rho_s^i. \tag{2.16}$$

For useful applications it is necessary to combine fermionic path integrals with ordinary bosonic path integrals using Wiener measure and Brownian motion. To do this one must take the approach (considered by Bochner [19]) where a probability measure is recovered (via the Kolmogorov extension theorem) from its finite-dimensional distributions. Thus one can obtain super Wiener measure by defining a measure (in a generalized sense) on the (m, n)-dimensional super Wiener space $\mathbb{R}_L^{(m,2n)[I]}$ of paths in superspace by defining finite distributions on $\mathbb{R}_L^{(m,2n)[J]}$ (where J is a finite subset of I containing N elements) to be

$$\begin{aligned} &F_J(\underline{x}^1, \ldots, \underline{x}^N, \underline{\theta}^1, \ldots, \underline{\theta}^N, \underline{\rho}^1, \ldots, \underline{\rho}^N) \\ &\qquad = f_J(\underline{x}^1, \ldots, \underline{x}^N)\phi_J(\underline{\theta}^1, \ldots, \underline{\theta}^N, \underline{\rho}^1, \ldots, \underline{\rho}^N) \end{aligned} \tag{2.17}$$

where

$$f_J(\underline{x}^1, \ldots, \underline{x}^N) = P_{t^1}(\underline{0}, \underline{x}^1) \ldots P_{t^N - t^{N-1}}(\underline{x}^{N-1}, \underline{x}^N)$$

with

$$P_t(\underline{x}, \underline{y}) = \left(\frac{2\pi}{t}\right)^{m/2} \exp\left(\frac{-(\underline{x} - \underline{y})^2}{2t}\right) \tag{2.18}$$

and

$$\begin{aligned} &\phi_J(\underline{\theta}^1, \ldots, \underline{\theta}^N, \underline{\rho}^1, \ldots, \underline{\rho}^N) \\ &\qquad = \exp[-\mathrm{i}(\underline{\rho}^1 \cdot \underline{\theta}^1 + \underline{\rho}^2 \cdot (\underline{\theta}^2 - \underline{\theta}^1) + \ldots + \underline{\rho}^N \cdot (\underline{\theta}^N - \underline{\theta}^{N-1}))]. \end{aligned} \tag{2.19}$$

This measure is called super Wiener measure, and is denoted $\mathrm{d}\mu_s$. Clearly $\mathrm{d}\mu_s$ is simply the product of standard Wiener measure $\mathrm{d}\mu_b$ with the fermionic Wiener measure $\mathrm{d}\mu_f$ defined above. A random variable is again a sequence of pairs of functions and sets (G_N, J_N), in this case such that $\int \mathrm{d}\mu_f\, G_N$ converges in L^2 to an L^2 random variable on bosonic Wiener space.

Grassmann algebra elements are not used to model real situations directly; their use is motivated by the algebraic properties of the function spaces of these variables. Applications usually give real or complex results by the use of Berezin integration. In the case of fermionic paths, applications always involve calculating the expectations of random variables, and thus it is sufficient to consider random variables to be equal if they have equal expectations, and such equality between random variables will be indicated by $=_I$. In the next section a particular class of random variables on super Wiener space, known as stochastic integrals, will be defined.

One also needs the concept of an adapted stochastic process on super Wiener space. The usual definition of adapted cannot be generalized in a straightforward way because here one is not working directly with σ algebras of measurable sets.

Definition 2.2. (a) Let t be a positive number and let $\{F_s : 0 \leq s \leq t\}$ be a collection of random variables on $\mathbb{R}_L^{(m,2n)[[0,t]]}$. Then F_s is said to be a *stochastic process* on this space. (b) Suppose that for each $s \in I = [0,t]$ the random variable F_s corresponds to the sequence of pairs $(J_{s,M}, F_{s,M} : M = 1, 2, \ldots)$. Then the stochastic process F_s is said to be adapted if for all positive integers M and each $s \in I$

$$J_{s,M} \subset [0, s]. \tag{2.20}$$

Berezin integration does not define a true measure, and thus one may not be able to prove results on Grassmann Wiener space using standard analytic techniques. Some useful tools for handling analytic questions are the norms which will now be defined, together with a lemma relating fermionic Wiener integrals to purely bosonic integrals.

Definition 2.3. Let J be a finite subset of I containing N elements, and let $f \in L^{2\prime}(\mathbb{R}_L^{(m,n)[J]}, \mathbb{C})$ with

$$f(\underline{x}^1, \ldots, \underline{x}^N, \underline{\theta}^1, \ldots, \underline{\theta}^N, \underline{\rho}^1, \ldots, \underline{\rho}^N) = \sum_{\mu \in M_{nN}} \sum_{\nu \in M_{nN}} f_\mu^\nu(\underline{x}^1, \ldots, \underline{x}^N) \theta^\mu \rho^\nu.$$

(a) The function $|f|_1 \in L^2(\mathbb{R}^{mN}, \mathbb{R})$ is defined by

$$|f|_1(\underline{x}^1, \ldots, \underline{x}^N) = \sum_{\mu \in M_{nN}} \sum_{\nu \in M_{nN}} |f_\mu^\nu(\underline{x}^1, \ldots, \underline{x}^N)|. \tag{2.21}$$

(b) The function $|f|_2 \in L^1(\mathbb{R}^{mN}, \mathbb{R})$ is defined by

$$\left(|f|_2(\underline{x}^1, \ldots, \underline{x}^N)\right)^2 = \sum_{\mu \in M_{nN}} \sum_{\nu \in M_{nN}} |f_\mu^\nu(\underline{x}^1, \ldots, \underline{x}^N)|^2. \tag{2.22}$$

Lemma 2.4. Suppose that G is a random variable on super Wiener space, defined finitely on $J \subset I$. Then

$$|\mathbb{E}_s(G)| \leq \mathbb{E}_b(|G|_1) \tag{2.23}$$

where $\mathbb{E}_b$ denotes expectations with respect to bosonic Wiener measure.

Proof. Suppose that the set J contains N elements and that

$$G(\underline{x}^1,\ldots,\underline{x}^N,\underline{\theta}^1,\ldots,\underline{\theta}^N,\underline{\rho}^1,\ldots,\underline{\rho}^N)=\sum_{\mu\in M_{nN}}\sum_{\nu\in M_{nN}}G^\nu_\mu(\underline{x}^1,\ldots,\underline{x}^N)\theta^\mu\rho_\nu. \tag{2.24}$$

Then

$$\begin{aligned}|\mathbb{E}_b(G)| &\le \sum_{\mu\in M_{nN}}\sum_{\nu\in M_{nN}}\Bigg|\int \mathrm{d}x^{mN}\,\mathrm{d}\theta^{nN}\,\mathrm{d}\rho^{nN}f_J(\underline{x}^1,\ldots,\underline{x}^N) \\ &\qquad \times \phi_J(\underline{\theta}^1,\ldots,\underline{\theta}^N,\underline{\rho}^1,\ldots,\underline{\rho}^N)G^\nu_\mu(\underline{x}^1,\ldots,\underline{x}^N)\theta^\mu\rho_\nu\Bigg| \\ &\le \sum_{\mu\in M_{nN}}\sum_{\nu\in M_{nN}}\left|\int \mathrm{d}x^{mN}f_J(\underline{x}^1,\ldots,\underline{x}^N)G^\nu_\mu(\underline{x}^1,\ldots,\underline{x}^N)\right| \end{aligned}\tag{2.25}$$

because, if α_i, $i=1,\ldots,N$, are even Grassmann elements whose squares are zero, the Taylor expansion of $\exp(\sum_{i=1}^N \alpha_i)$ contains each non-zero term with coefficient exactly 1. Thus

$$|\mathbb{E}_s(G)|\le \sum_{\mu\in M_{nN}}\int \mathrm{d}x^{mN}\sum_{\nu\in M_{nN}}f_J(\underline{x}^1,\ldots,\underline{x}^N)|G^\nu_\mu(\underline{x}^1,\ldots,\underline{x}^N)| = \mathbb{E}_b\left(|G|_1\right). \tag{2.26}$$

3. Bosonic and fermionic stochastic integration

A key theorem in stochastic calculus is the general Itô theorem, which gives the chain rule for stochastic differentiation. This differs from the equivalent result for a smooth deterministic path by including a second-order term, the Itô correction. The purpose of this section is to extend this result to include Grassmann Brownian motion. The inclusion of the fermionic sector does not involve any additional stochastic correction term in the Itô formula because Grassmann Brownian motion is based on a zero free Hamiltonian [2].

The Itô formula for m-dimensional Brownian motion, which generalizes the fundamental theorem of calculus, states that

$$f(b_t,t)-f(b_0,0)=\int_0^t\sum_{i=1}^m\partial_i f(b_s,s)\,\mathrm{d}b^i_s+\int_0^t\partial_s f(b_s,s)\,\mathrm{d}s+\tfrac{1}{2}\int_0^t\sum_{i=1}^m\partial_i\partial_i f(b_s)\,\mathrm{d}s \tag{3.1}$$

where f is a suitably well-behaved function of $\mathbb{R}^m\times\mathbb{R}^+$. This result is a special case of the general Itô formula. In the case of purely fermionic Brownian motion one has the simple result that

$$f(\theta_t,t)-f(\theta_0,0)=_I\int_0^t\partial_s f(\theta_s,s)\mathrm{d}s \tag{3.2}$$

where the notation $=_I$ means that one has two random variables on $\mathbb{R}_L^{(0,2n)[(0,t)]}$ with equal expectations. (A definition of the expression on the right-hand side of equation (3.2) is given below.) This suggests that one should simply consider $\int f(\theta_s, s)\,\mathrm{d}\theta_s$ to be zero, and thus that generalizations of the classical Itô formula to include fermionic paths will not involve integrals along fermionic paths, as indeed will be the case. A simple definition of time integral will now be given, while theorem 3.3 establishes the existence of the Itô integral of a large class of adapted processes on super Wiener space. The key theorem 3.5 provides the corresponding generalization of the Itô theorem.

The absence of the $\mathrm{d}\theta$ and $\mathrm{d}\rho$ terms may be made clearer by the following observations. First, because fermion paths are defined in phase space, with the analogue of both ps and qs present as θs and ρs, ρ is in a sense equal to $\mathrm{d}\theta$—indeed the formalism might be developed along these lines. However, if one is applying these methods to diffusion theory, such an approach is superfluous because one can already handle all combinations of the derivative operators $\partial/\partial\theta_i$ directly (in contrast to the bosonic case where first- and second-order differential operators other than the flat Laplacian can only be handled using stochastic calculus, and where operators of degree higher than two cannot be handled at all). Before turning to the Itô integral, a simple integral with respect to time is needed.

Definition 3.1. Let F_s be a $[0,t]$-adapted process on the super Wiener space $\mathbb{R}_L^{(m,2n)[[0,t]]}$. Suppose that, for each s in I, F_s corresponds (as in definition 2.2) to the sequence of pairs of subsets and functions $(J_{s,M}, F_{s,M} : M = 1,2,\ldots)$. For each integer $M = 1,2,\ldots$ let

$$J_M = \bigcup_{r=1}^{2^M-1} J_{t_r,M} \tag{3.3}$$

where, for $r = 1,\ldots,2^M-1$, $t_r = rt/2^M$. Also define functions K_M on $\mathbb{R}_L^{(m,2n)[J_M]}$ by

$$K_M = \left(\sum_{r=1}^{2^M-1} F_{t_r,M}\frac{t}{2^M}\right). \tag{3.4}$$

Then, if $\mathbb{E}_s(K_M)$ tends to a limit as M tends to infinity, the sequence (J_M, K_M) defines a Grassmann random variable which is denoted $\int_0^t F_s\,\mathrm{d}s$, and one says that F_s has a time integral. Also, if $0 < u < t$ and M is a positive integer, let $p(M,u)$ be the greatest integer such that $p(M,u)t/2^M < u$, and set

$$L_M^u = \sum_{r=1}^{p(M,u)} F_{t_r,M}\frac{t}{2^M}. \tag{3.5}$$

Then, if $\mathbb{E}_s(L_M^u)$ tends to a limit as M tends to infinity, the sequence of pairs $(J_M, L_M^u : M = 1,2,\ldots)$ defines a Grassmann random variable which is denoted $\int_0^u F_s\,ds$. (This definition may also be applied to generalized random variables.)

Definition 3.2. (a) $M_\Omega^{1\prime}[0,t]$ denotes the set of all $[0,t]$-adapted processes F_s on the super Wiener space $\mathbb{R}_L^{(m,2n)[[0,t]]}$ such that (i) $\int_0^t |F_s|_1\,\mathrm{d}s$ exists and is finite, (ii) $\mathbb{E}_b(|F_s|_1)$ is bounded on $[0,t]$. (b) $M_\Omega^{2\prime}[0,t]$ denotes the set of all $[0,t]$-adapted processes F_s on the super Wiener space $\mathbb{R}_L^{(m,2n)[[0,t]]}$ such that (i) $\int_0^t |F_s|_2^2\,\mathrm{d}s$ exists and is finite, (ii) $\mathbb{E}_b\left(|F_s|_2^2\right)$ is bounded on $[0,t]$.

Note that in this definition, if F_s is defined by the sequence $(J_{s,M}, F_{s,M} : M = 1,2,\ldots)$ (following definitions 2.1 and 2.2), $|F_s|_1$ is defined by the sequence $(J_{s,M}, |F_{s,M}|_1 : M = 1,2,\ldots)$. $|F_s|_1$ may be a generalized random variable on Wiener space. A similar approach is taken to $|F_s|_2$. Also, following definition 3.1, the time integrals are Riemann integrals.

Theorem 3.3. (a) For $a = 1,\ldots,m$ suppose that $F^a \in M_\Omega^{2\prime}[0,t]$. For each $M = 1,2,\ldots$ let $J_M = \{t_1,\ldots,t_{2^M-1}\}$ with $t_r = rt/2^M$ for $r = 1,\ldots,2^M-1$. Also let $G_M \in L^{2\prime}(\mathbb{R}_L^{(m,2n)[J_M]}, \mathbb{C})$ with

$$G_M(\underline{x}^1,\ldots,\underline{x}^{2^M-1},\underline{\theta}^1,\ldots,\underline{\theta}^{2^M-1},\underline{\rho}^1,\ldots,\underline{\rho}^{2^M-1})$$

$$= \sum_{a=1}^{m}\sum_{r=1}^{2^M-1} F^{M,a}(\underline{x}^1,\ldots,\underline{x}^r,\underline{\theta}^1,\ldots,\underline{\theta}^r,\underline{\rho}^1,\ldots,\underline{\rho}^r)(x^{r+1,a}-x^{r,a}). \quad (3.6)$$

Then the sequence (J_M, G_M) defines a Grassmann random variable which is denoted

$$\int_0^t \sum_{a=1}^{m} F_s^a \, \mathrm{d}b_s^a .$$

Also $|\int_0^t \sum_{a=1}^m F_s^a \, \mathrm{d}b_s^a|_2^2$ is a random variable and

$$\mathbb{E}\left(|\int_0^t \sum_{a=1}^{m} F_s^a \, \mathrm{d}b_s^a|_2^2 \right) = \mathbb{E}\left(\int_0^t \sum_{a=1}^{m} |F_s^a|_2^2 \, \mathrm{d}s \right). \quad (3.7)$$

(Integrals over the interval $(0,u)$, where $0 < u < t$, are defined by a similar modification to that made in definition 3.1.)

Proof. $\int \mathrm{d}\mu_f(G_M)$ converges to a random variable on m-dimensional Wiener space, and thus (J_M, G_M) defines a super random variable. Using the fact that, if $q < r$,

$$F_{t_r,M\,\mu\nu}(\underline{x}^1,\ldots,\underline{x}^r)F_{t_q,M\,\mu\nu}(\underline{x}^1,\ldots,\underline{x}^q)(x^{q+1}-x^q)$$

is independent of $(x^{r+1}-x^r)$, and the standard result that

$$\mathbb{E}_b((b_t-b_s)(b_t-b_s)) = |s-t| \quad (3.8)$$

one obtains

$$\mathbb{E}_b\left(|G_M|_2^2\right) = \mathbb{E}_b\left(\sum_{r=1}^{2^M-1} |F_{t_r,M}|_2^2 (t_{r+1}-t_r)\right) \quad (3.9)$$

which tends to the correct limit as M tends to infinity since F is in $M_\Omega^{2\prime}[0,t]$.

More generally it is useful to combine both kinds of integration, obtaining the following definition of a stochastic integral.

Definition 3.4. Suppose that t is a positive real number and that Z_s is a stochastic process on the super Wiener space $\mathbb{R}_L^{(m,2n)[[0,t]]}$ such that, if $0 \le t_1 \le t_2 \le t$,

$$Z_{t_1} - Z_{t_2} = \int_{t_1}^{t_2} A_s \, \mathrm{d}s + \int_{t_1}^{t_2} \sum_{a=1}^{m} C_s^a \mathrm{d}b_s^a \tag{3.10}$$

where A_s and C_s^a are $[0,t]$-adapted processes on the super Wiener space, with $A_s \in M_\Omega^{1\prime}[0,t]$ and $C_s^a \in M_\Omega^{2\prime}[0,t]$. Then Z_s is said to be a *stochastic integral.*

The key theorem, underpinning much of this paper, is the following generalized Itô theorem.

Theorem 3.5. Suppose that Z_s^j ($j = 1, \ldots, p+q$) are stochastic integrals on the super Wiener space $\mathbb{R}_L^{(m,2n)[(0,t)]}$ with

$$Z_s^j - Z_0^j = \int_0^t A_s^j \, \mathrm{d}t + \int_0^t \sum_{a=1}^{m} C_s^{aj} \, \mathrm{d}b^a(s) \tag{3.11}$$

and that Z^j is even for $j = 1, \ldots, p$ and odd for $j = p+1, \ldots, p+q$. Also suppose that $\mathbb{E}(A_s^j)^2$ is bounded for each s in $[0,t]$. Then, if $H \in C^{5\prime}(\mathbb{R}^{(p,q)}, \mathbb{C})$ and $Z_{s,M}^j$ denotes the Mth term of the sequence defining the random variable Z_s^j (following definitions 3.1 and 3.2), the sequence $H(Z_{s,M})$ defines a stochastic process denoted H_s, and H_s is also a stochastic integral with

$$\begin{aligned} H_s - H_0 =_I & \int_0^t \sum_{j=1}^{p+q} \sum_{a=1}^{m} \partial_j H(Z_s) \, C_s^{aj} \, \mathrm{d}b_s^a \\ & + \int_0^t \left(\sum_{j=1}^{p+q} A_s^j \partial_j H(Z_s) + \tfrac{1}{2} \sum_{a=1}^{m} \sum_{j=1}^{p+q} \sum_{k=1}^{p+q} C_s^{ak} C_s^{aj} \, \partial_j \partial_k H(Z_s) \right) \mathrm{d}s. \end{aligned} \tag{3.12}$$

(Here the function H is extended to even Grassmann variables using the truncated Taylor series of equation (2.4).)

Proof. Clearly

$$H(Z_{t,M}) = H(Z_{0,M}) + \sum_{r=1}^{2^M - 1} \left(H(Z_{t_{r+1},M}) - H(Z_{t_r,M}) \right). \tag{3.13}$$

Now

$$\begin{aligned} H(Z_{t_{r+1},M}) - H(Z_{t_r,M}) = & \sum_{j=1}^{p+q} \Delta^{rM}(Z^j) \partial_j H(Z_{t_r,M}) \\ & + \tfrac{1}{2} \sum_{j=1}^{p+q} \sum_{k=1}^{p+q} \Delta^{rM}(Z^j) \Delta^{rM}(Z^k) \partial_j \partial_k H(Z_{t_r,M}) + R_r \end{aligned} \tag{3.14}$$

where

$$\Delta^{Mr}(Z^j) = Z^j_{t_{r+1},M} - Z^j_{t_r,M} \tag{3.15}$$

and R_r is a remainder term to be analysed. By standard arguments ([10], p 31) it is sufficient to consider the case where H and its derivatives up to fifth order are bounded functions. Also, since H is extended from $\mathbb{R}^{p,q}$ to $\mathbb{B}^p_{L,0} \times \mathbb{B}^q_{L,1}$ by the truncated Taylor series (2.2), H obeys a truncated Taylor's theorem with remainder at the third order. Thus one finds as in the classical case [10] that $|R_r| \leq k/2^{2M}$ for some k independent of M, and hence, using lemma 2.4,

$$\begin{aligned}\mathbb{E}\left(H(Z_{t,M}) - H(Z_{0,M})\right) = \mathbb{E}\Bigg(\sum_{r=1}^{2^M-1} \Bigg(\sum_{j=1}^{p+q} \Delta^{rM}(Z^j)\partial_j H(Z_{t_r,M}) \\ + \tfrac{1}{2} \sum_{j=1}^{p+q} \sum_{k=1}^{p+q} \Delta^{rM}(Z^j)\Delta^{rM}(Z^k)\partial_j\partial_k H(Z_{t_r,M}) \Bigg) \Bigg) + R'_M\end{aligned}$$

where $|R'_M| < K/2^M$ for some K independent of M. The result then follows on taking limits as M tends to infinity.

4. Supersymmetric Itô formulae

In this section the results of the previous section are used to derive some supersymmetric Itô formulae for random variables on (m,m)-dimensional super Wiener space. These formulae are stochastic versions of a theory of calculus on a (1,1)-dimensional superspace (parametrized by a real variable t and an odd Grassmann variable τ), which will now be described. First, the superderivative D_T is defined to be the operator

$$D_T = \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial t}. \tag{4.1}$$

This operator may act on a function of the form $F(t,\tau) = A(t) + \tau B(t)$ where $A(t)$ has a time derivative. (As the notation implies, A and B may have either Grassmann parity.) One then has

$$D_T F(t,\tau) = B(t) + \tau \frac{\partial A}{\partial t}(t). \tag{4.2}$$

If $B(t)$ is also differentiable, an easy calculation then shows that $D_T^2 = \partial/\partial t$, and so in systems where the Hamiltonian H is the square of a supercharge Q which is an odd operator the imaginary-time Schrödinger equation

$$\frac{\partial f}{\partial t} = -Hf \tag{4.3}$$

has a square root [20]

$$D_T f = Qf. \tag{4.4}$$

(This equation implies the previous one, since Q is an odd operator and thus anticommutes with D_T.) Noting that

$$\exp(-Ht - Q\tau) = \exp(-Ht)(1 - Q\tau) \tag{4.5}$$

one may check by direct calculation that, provided H is suitably regular,

$$D_T\big(\exp(-Ht - Q\tau)\big) = Q\exp(-Ht - Q\tau). \tag{4.6}$$

Thus $\exp(-Ht - Q\tau)$ is the evolution operator for the square root of the Schrödinger equation. Also, since $Q\tau = -\tau Q$

$$Q\exp(-Ht - Q\tau) = \big(\exp(-Ht - Q\tau)\big)\,(Q - 2\tau H)$$

and thus

$$D_T\big(\exp(-Ht - Q\tau)\big) = \big(\exp(-Ht - Q\tau)\big)\,(Q - 2\tau H) \tag{4.7}$$

a result that will be useful in section 6.

Turning now to integration, one may define an integral with respect to a (1,1)-dimensional variable $S = (s, \sigma)$, including both odd and even limits, in the following way [4]:

$$\int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, F(s,\sigma) =_{\mathrm{def}} \int \mathrm{d}\sigma \int_0^{t+\sigma\tau} \mathrm{d}s\, F(s,\sigma). \tag{4.8}$$

The even integral on the right-hand side of this equation is evaluated by regarding $\int_0^u \mathrm{d}s\, F(s,\sigma)$ as a function of u, and evaluating this function when $u = t + \sigma\tau$ by Taylor expansion about $t = u$. Integration with respect to the odd variable σ is then carried out according to the usual Berezin prescription. Explicitly if $F(t,\tau) = A(t) + \tau B(t)$ one has

$$\int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, F(s,\sigma) = \tau A(t) + \int_0^t B(s)\mathrm{d}s. \tag{4.9}$$

This definition of integration leads to a supersymmetric version of the fundamental theorem of calculus in the form

$$\int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, D_S F(s,\sigma) = F(t,\tau) - F(0,0) \tag{4.10}$$

and hence a natural rule for the superderivative of such an integral with respect to its upper limits is obtained as

$$D_T \int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, G(s,\sigma) = G(t,\tau). \tag{4.11}$$

Reverting to stochastic calculus, one may now derive a supersymmetric version of both the restricted and the general Itô formulae. These are presented in turn as theorems 4.2 and 4.3. To begin with, a time integral for superpaths must be defined.

Definition 4.1. Suppose that G_s and H_s are adapted stochastic processes on super Wiener space, and that $H_s \in M_\Omega^{1\prime}[0,t]$. Then, if $F_S = G_s + \sigma H_s$,

$$\int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, F_S =_{\mathrm{def}} \int_0^t \mathrm{d}s\, H_s + \tau G_t. \tag{4.12}$$

(Throughout the remainder of this paper the summation convention, that repeated indices are to be summed over their range, will be adopted.) The following theorem gives a supersymmetric version of the Itô theorem. The superpaths Z_S and ζ_S defined below are the stochastic version of the standard superfields used in supersymmetric quantum mechanics.

Theorem 4.2. For $i = 1, \ldots, p$ and $a = 1, \ldots, m$, let x_s^i and $f_{(s)a}^i$ be adapted stochastic processes on (m, m)-dimensional super Wiener space, and ϕ be a function in $C^{5\prime}(\mathbb{R}^{p,m})$ which is linear in the odd argument, and let $(\underline{x}, \underline{\theta}) \in \mathbb{R}_L^{(p,m)}$. Also, for any $g \in C^{5\prime}(\mathbb{R})$ let

$$g_{t,\tau}(x,\theta) =_{\mathrm{def}} g\left(\int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, \phi\left(x + z_S + \frac{\mathrm{i}}{\sqrt{2}}\sigma\eta_s, \theta + \zeta_S\right)\right) \tag{4.13}$$

where, recalling from equation (2.16) that $\xi_s^a = \theta_s^a - \mathrm{i}\rho_s^a$,

$$z_S^i = x_s^i + \frac{\mathrm{i}}{\sqrt{2}}\sigma\xi_s^a f_{(s)a}^i \qquad \zeta_S^a = \xi_s^a + \sqrt{2}\mathrm{i}\sigma\dot{b}_s^a \qquad \eta_s^i = \theta^a f_{(s)a}^i. \tag{4.14}$$

(Here the notation $\dot{b}_s$ is formal; it is to be interpreted in combination with $\mathrm{d}s$ as $\dot{b}_s\,\mathrm{d}s = \mathrm{d}b_s$. The placing of ζ_S^a in formulae will be such that only this combination will occur.) Also ϕ has been extended to even Grassmann elements as in equation (2.2), and it has been assumed that the canonical projection of x_s is always real. (The theorem is also valid when x is complex and ϕ analytic.) Then

$$\begin{aligned} g_{t,\tau}(x,\theta) - g_{0,0} =_I \int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s \Bigg\{ & g'_{s,\sigma}(x,\theta)\,\phi\left(x + z_S + \frac{\mathrm{i}}{\sqrt{2}}\sigma\eta_s, \theta + \zeta_S\right) \\ & - \left[2\sigma g''_{s,\sigma}(x,\theta)\,\delta_a\phi\left(x + z_S + \frac{\mathrm{i}}{\sqrt{2}}\sigma\eta_s, \theta + \zeta_S\right)\right. \\ & \left.\left. \times\, \delta_a\phi\left(x + z_S + \frac{\mathrm{i}}{\sqrt{2}}\sigma\eta_s, \theta + \zeta_S\right)\right]\right\}. \end{aligned} \tag{4.15}$$

Proof.

$$\begin{aligned} g_{t,\tau}(x,\theta) - g_{0,0}(x,\theta) &= g_{t,0}(x,\theta) + \tau\phi(x + x_t, \theta + \xi_t)g'_{t,0} - g_{0,0} \\ &= \tau\phi(x + x_t, \theta + \xi_t)g'_{t,0} + g\left(\int_0^t \mathrm{d}s\left(\frac{\mathrm{i}}{\sqrt{2}}(\theta^a + \xi_s^a)f_{(s)a}^i\right.\right. \\ &\quad \left.\left. \times\, \partial_i\phi(x + x_s, \theta + \xi_s)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&+\sqrt{2}\mathrm{i}\int_0^t \mathrm{d}b_s^a \delta_a \phi(x+x_s,\theta+\xi_s)\Big) - g_{0,0}(x,\theta) \\
=_I\ & \tau\phi(x+x_t,\theta+\xi_t){g'}_{t,0}(x,\theta) \\
&+\int_0^t \mathrm{d}s\bigg(\frac{\mathrm{i}}{\sqrt{2}}g'_{s,0}(x,\theta)\,(\theta^a+\xi_s^a)f^i_{(s)a}\partial_i\phi(x+x_s,\theta+\xi_s) \\
&-g''_{s,0},\delta_a\phi(x+x_s,\theta+\xi_s)\delta_a\phi(x+x_s,\theta+\xi_s)\bigg) \\
&+\sqrt{2}\mathrm{i}\int_0^t \mathrm{d}b_s^a g'_{0,s}(x,\theta)\,\delta_a\phi(x+x_s,\theta+\xi_s).
\end{aligned}
\tag{4.16}
$$

Also

$$
\begin{aligned}
&\int_0^\tau \mathrm{d}\sigma\int_0^t \mathrm{d}s\, g'_{s,\sigma}(x,\theta)\,\phi\left(x+z_S+\frac{\mathrm{i}}{\sqrt{2}}\sigma\eta_s,\theta+\zeta_S\right) \\
&\quad=_I \tau\phi(x+x_t,\theta+\xi_t)g'_{t,0}(x,\theta) \\
&\quad+\frac{\mathrm{i}}{\sqrt{2}}\int_0^t \mathrm{d}s\, g'_{0,s}(x,\theta)\,(\theta^a+\xi_s^a)f^i_{(s)a}\partial_i\phi(x+x_s,\theta+\xi_s) \\
&\quad+\sqrt{2}\mathrm{i}\int_0^t \mathrm{d}b_s^a\, g'_{0,s}(x,\theta)\,\delta_a\phi(x+x_s,\theta+\xi_s) \\
&\quad+\int_0^t \mathrm{d}s\,\phi(x+x_s,\theta+\xi_{s-})g''_{0,s}(x,\theta)\,\phi(x+x_s,\theta+\xi_s)
\end{aligned}
\tag{4.17}
$$

using theorem 3.5. The result now follows immediately from the fact that

$$
\begin{aligned}
&\int_0^t \mathrm{d}s\,\phi(x+x_s,\theta+\xi_{s-})\phi(x+x_s,\theta+\xi_s)g''_{0,s}(x,\theta) \\
&\qquad=_I \int_0^t \mathrm{d}s\, g''_{s,0}(x,\theta)\sum_{a=1}^n \delta_a\phi(x+x_s,\theta+\xi_s)\delta_a\phi(x+x_s,\theta+\xi_s).
\end{aligned}
\tag{4.18}
$$

A second theorem, which is a supersymmetric version of the restricted Itô theorem, will now be proved.

Theorem 4.3. Suppose that $f\in C^{S\prime}(\mathbb{R}^{p,m})$. Also, for $i=1,\ldots,p$ let x_s^i be an adapted process such that

$$
x_s^i - x_0^i = \int_0^t e_a^i(x_s)\mathrm{d}b_s^a + \int_0^t L_s^i\,\mathrm{d}s
\tag{4.19}
$$

where, for $i=1,\ldots,p$ and $a=1,\ldots,m$, e_a^i is a function on $\mathbb{R}^p$ and L_s^i is an adapted stochastic process on (m,m)-dimensional super Wiener space. (It will be shown in the next section that, under reasonable assumptions on e_a^i and L_s^i, processes satisfying equations such as (4.17) do exist.) Then, using the notation of the preceding theorem,

$$f\left(x+z_T+\frac{\mathrm{i}}{\sqrt{2}}\tau\eta,\theta+\theta_t\right)-f(x,\theta)$$

$$=_I \int_0^\tau \mathrm{d}\sigma\int_0^t \mathrm{d}s\left[\frac{\mathrm{i}}{\sqrt{2}}\psi^a e_a^i(x+x_s)\partial_i f\left(z_S+x+\frac{\mathrm{i}}{\sqrt{2}}\sigma\eta,\theta+\theta_s\right)\right.$$

$$\left.+\sigma e_a^i(x_s+x)e_a^j(x_s+x)\partial_i\partial_j f\left(z_S+x+\frac{\mathrm{i}}{\sqrt{2}}\sigma\eta,\theta+\theta_s\right)\right]$$

$$+\int_0^\tau \mathrm{d}\sigma\int_0^t dx_s^i\,\sigma\partial_i f\left(z_S+x+\frac{\mathrm{i}}{\sqrt{2}}\sigma\eta,\theta+\theta_s\right). \tag{4.20}$$

Proof. Using the Itô formula (theorem 3.5),

$$f\left(x+z_T+\frac{\mathrm{i}}{\sqrt{2}}\tau\eta_t,\theta+\theta_t\right)-f(x,\theta)$$

$$=_I \frac{\mathrm{i}}{\sqrt{2}}\tau(\theta^a+\xi_t^a)e_a^i(x_t+x)\partial_i f(x+x_t,\theta_t+\theta)$$

$$+\int_0^\tau \mathrm{d}\sigma\int_0^t \mathrm{d}x_s^i\,\sigma\partial_i f(x+x_s,\theta+\theta_s)$$

$$+\tfrac{1}{2}\int_0^\tau \mathrm{d}\sigma\int_0^t \mathrm{d}s\,\sigma e_a^i(x+x_s)e_a^j(x+x_s)\partial_i\partial_j f(x+x_s,\theta+\theta_s). \tag{4.21}$$

Also

$$\int_0^\tau\int_0^t \mathrm{d}s\,\frac{\mathrm{i}}{\sqrt{2}}\psi^a e_a^i(x+x_s)\partial_i f\left(z_S+x+\frac{\mathrm{i}}{\sqrt{2}}\sigma\eta,\theta+\theta_s\right).$$

$$=_I \frac{\mathrm{i}}{\sqrt{2}}\tau(\theta^a+\xi_t^a)e_a^i(x_t+x)\partial_i f(x+x_t,\theta_t+\theta)$$

$$-\tfrac{1}{2}\int_0^t \mathrm{d}s\,(\theta^b+\xi_s^b)e_b^j(x+x_s)\psi^a e_a^i(x+x_s)\partial_i\partial_j f(x_s+x,\theta+\theta_s)$$

$$=_I \frac{\mathrm{i}}{\sqrt{2}}\tau(\theta^a+\xi_t^a)e_a^i(x_t+x)\partial_i f(x+x_t,\theta_t+\theta)$$

$$-\tfrac{1}{2}\int_0^\tau\int_0^t \mathrm{d}s\,\sigma e_b^j(x+x_s)e_a^i(x+x_s)$$

$$\times\,\partial_i\partial_j f\left(y_S+x+\frac{\mathrm{i}}{\sqrt{2}}\sigma\eta,\theta+\theta_s\right). \tag{4.22}$$

The result now follows.

These theorems are crucial for the Feynman–Kac formula to be developed in the final section. Before proving this result a further technical step is required: it must be shown that a reasonable class of stochastic differential equations can be solved. This is the subject of the next section.

5. Stochastic differential equations

In order to obtain generalized Feynman–Kac formulae for fermionic systems in curved space, it is necessary to consider more general bosonic stochastic processes than Brownian motion; the required processes are in fact solutions of stochastic differential equations, and in this section the usual theory of stochastic differential equations will be extended to include fermionic Brownian motion. The key theorem in this section establishes the existence of unique solutions to a wide class of stochastic differential equations; in the following section functions of solutions to such equations are analysed using the supersymmetric Itô formulae of the preceding section to obtain a supersymmetric Feynman–Kac formulae. The main theorem of this section establishes that differential equations of the form

$$\mathrm{d}Z^j_s = A^j_a(Z_s,\theta_s,\rho_s,s)\,\mathrm{d}b^a_s + B^j(Z_s,\theta_s,\rho_s,s)\,\mathrm{d}s \tag{5.1}$$

have unique solutions, provided that the functions A^j_i and B^j are sufficiently regular. In the following section processes which are solutions of suitably chosen stochastic differential equations will be used to construct Feynman–Kac formulae for various differential operators.

To begin with it is necessary to explain precisely what is meant by a stochastic differential equation of the form (5.1).

Definition 5.1. Let p, m and n be positive integers. For $j = 1,\dots,p$ and $a = 1,\dots,m$ let A^j_a and B^j map $\mathbb{B}^p_{L,0} \times \mathbb{B}^{2n}_{L,1} \times \mathbb{R}^+$ into $\mathbb{B}_{L0}$. Also let A be a fixed element of $\mathbb{C}^p$. Then, if, for $j = 1,\dots,p$, Z^j_s is a stochastic process in $M^{2\prime}_\Omega[0,t]$ such that

$$Z^j_s - Z^j_0 = \int_0^t A^j_a(Z_s,\theta_s,\rho_s,s)\,\mathrm{d}b^a_s + \int_0^t B^j(Z_s,\theta_s,\rho_s,s)\,\mathrm{d}s \tag{5.2a}$$

$$Z_0 = A \tag{5.2b}$$

(where (b_s,θ_s,ρ_s) are Brownian paths in (m,n)-dimensional super Wiener space), it is said that Z_s satisfies the *stochastic differential equation* (5.2a) with initial condition (5.2b). This definition implies that the $B^j(Z_s,\theta_s,\rho_s,s)$ are in $M^{1\prime}_\Omega[0,t]$ and the $A^j_a(Z_s,\theta_s,\rho_s,s)$ are in $M^{2\prime}_\Omega[0,t]$.

In order to prove that such equations have solutions, and that the solutions are unique, one essentially follows the classical method of proof [20].

Theorem 5.2. With the notation of definition 5.1, suppose that, for all x and y in $\mathbb{B}^{p,0}_L$, all s in $[0,t]$ and all (μ,ν) in $M_n \times M_n$, the coefficient functions of A^j_a and B^j satisfy

$$\begin{aligned}
&|A^{j\mu}_{a\nu}(x,s) - A^{j\mu}_{a\nu}(y,s)| \le k|x-y|/(pm \times 2^{2n}) \\
&|A^{j\mu}_{a\nu}(x,s)| \le k(1+|x|)/(pm \times 2^{2n}) \\
&|B^{j\mu}_{\nu}(x,s) - B^{j\mu}_{\nu}(y,s)| \le k|x-y|/(p \times 2^{2n}) \\
&|B^{j\mu}_{\nu}(x,s)| \le k(1+|x|)/(p \times 2^{2n})
\end{aligned} \tag{5.3}$$

for some fixed positive number k. Then there exists a unique solution to the stochastic differential equation

$$\begin{aligned} \mathrm{d}Z_s^j &= A_a^j(Z_s, \theta_s, \rho_s, s)\mathrm{d}b_s^a + B^j(Z_s, \theta_s, \rho_s, s)\mathrm{d}s \\ Z_0 &= A. \end{aligned} \tag{5.4}$$

(Two random variables F and G are said to be equal if $\mathbb{E}(|F-G|_2) = 0$.)

Proof. Suppose that x_s and y_s are two solutions to (5.4). Then, using theorem 3.3,

$$\begin{aligned} \mathbb{E}|x_s^j - y_s^j|_2^2 &\le 2s\int_0^s |B^j(x_u, \theta_u, \rho_u, u) - B^j(y_u, \theta_u, \rho_u, u)|_2^2\,\mathrm{d}u \\ &\quad + 2\int_0^s \sum_{i=1}^m |A_i^j(x_u, \theta_u, \rho_u, u) - A_i^j(y_u, \theta_u, \rho_u, u)|_2^2\,\mathrm{d}u \\ &\le 2k^2(1+s)\int_0^s |x_u - y_u|_2^2\,\mathrm{d}u. \end{aligned} \tag{5.5}$$

Hence $\mathbb{E}(|x_s - y_s|_2^2) = 0$ for all s in $[0,t]$, and thus any solution which exists is unique.

To establish existence of solutions consider the sequence $Z_{(r)s}^j$, $r = 1, 2, \ldots$, of stochastic processes on the super Wiener space $\mathbb{R}_L^{(m,2n)[[0,t]]}$ with

$$Z_{(0)s}^j = A$$

and, for $r > 0$,

$$Z_{(r)s}^j = A + \int_0^s B^j(Z_{(r-1)s}, \theta_u, \rho_u, u)\mathrm{d}u + \int_0^s A_a^j(Z_{(r-1)s}, \theta_u, \rho_u, u)\mathrm{d}b_s^a. \tag{5.6}$$

Then assume the following as inductive hypothesis, for all integers q up to and including some fixed integer r.

(a) $Z_{(r)s}^j$ is in $M_\Omega^{2\prime}[0,t]$;

(b)

$$\mathbb{E}\left(|Z_{(q)s}^j - Z_{(q-1)s}^j|_2^2\right) \le (Ms)^q/(q!) \tag{5.7}$$

where M is some positive constant.

Now $Z_{(1)s}^j$ is well defined and

$$\mathbb{E}|Z_{(1)s} - Z_{(0)s}|_2^2 \le 2|\int_0^s B^j(z_0, \theta_u, \rho_u, u)\mathrm{d}u|_2^2 + 2\int_0^s |A_i^j(z_0, \theta_u, \rho_u, u)|_2^2\mathrm{d}u. \tag{5.8}$$

Thus

$$\mathbb{E}|Z_{(1)s} - Z_{(0)s}|_2^2 \le 2k^2s^2 + k^2 s(1+|A|^2) \le Ms \tag{5.9}$$

if $M \ge 2k^2(t+1)(1+|A|)^2$.

For the inductive step, note that

$$\begin{aligned}\mathbb{E}|Z_{(r+1)s} - Z_{(r)s}|_2^2 &\le M \int_0^t \mathbb{E}|Z_{(r)s} - Z_{(r-1)s}|_2^2 \, \mathrm{d}s \\ &\le M \int_0^t \frac{(Ms)^r}{r!} \, \mathrm{d}s \\ &= \frac{(Mt)^{r+1}}{(r+1)!}.\end{aligned} \tag{5.10}$$

This implies that $Z_{(r+1)s}$ is in $M_\Omega^{2\prime}[0,t]$, and thus that the inductive hypothesis is satisfied. Thus the sequence $Z_{(r)s}$ converges in the required manner.

6. A supersymmetric Feynman–Kac formula

This section contains the main theorem, theorem 6.2, establishing a Feynman–Kac formula for a Hamiltonian H which is the square of supercharge operator Q, where Q is a Dirac-like operator (in a sense made precise below). Such Hamiltonians are referred to as supersymmetric, and the corresponding supersymmetric Feynman–Kac formula is given in terms of the supercharge operator Q rather than the Hamiltonian H. This is a technical advantage, because H is generally considerably more complicated than Q. It is also an advantage in principle to work directly with the fundamental Q rather than the derived H. To achieve this supersymmetric Feynman–Kac formula, the simple Feynman–Kac formula given in equation (1.2) has to be generalized in three ways. First, fermionic paths are included (and thus integrals are with respect to super Wiener measure). Secondly, ordinary Brownian paths b_t are replaced by solutions x_t of carefully chosen stochastic differential equations. Thirdly, and this is the vital step which leads to a supersymmetric Feynman–Kac formula, superpaths parametrized by a (1,1)-dimensional super variable are used.

Throughout this section (b_s, θ_s, ρ_s) denotes Brownian motion in (m,m)-dimensional super Wiener space. Also g is a C^5 Riemannian metric on $\mathbb{R}^m$ which has components g_{ij} in the natural coordinate system on $\mathbb{R}^m$, while the $m \times m$ matrices ${e^i}_a$ are the components of an orthonormal basis (e^a) of 1-forms on $\mathbb{R}^m$, so that

$$g_{ij}(x) = {e^i}_a(x){e^j}_a(x). \tag{6.1}$$

All derivatives of e^i_a up to fifth order are required to be uniformly bounded on R^m. The definition of the superpaths used in this section is essentially that used in section 4, but the choice of stochastic differential equation for the components x^i_s is determined by the metric.

Definition 6.1. Let x_s be the unique solution to the stochastic differential equation

$$\begin{aligned}\mathrm{d}x^i_s &= e^i_a(x+x_s)\mathrm{d}b^a_s + \tfrac{1}{2}(\delta^{ab} + \xi^a_s\xi^b_s){e^j}_a(x+x_s)\partial_j {e^i}_b(x+x_s)\mathrm{d}s \\ x^i_0 &= 0 \qquad i = 1, \ldots, m\end{aligned} \tag{6.2}$$

where e^i_a and $e^j_a \partial_j e^i{}_b$ are to be extended to even Grassmann elements by Taylor expansion truncated at the first-derivative term, so that (5.3) is maintained. Also, for $i, a = 1, \dots, m$ let

$$z^i_S = x^i_s + \frac{\mathrm{i}}{\sqrt{2}}\sigma \xi^a_s e^i{}_a(x + x_s) \qquad \zeta^a_S = \xi^a_s + \sqrt{2}\mathrm{i}\sigma \dot{b}^a_s \qquad \eta^i_s = \theta^a e^i{}_a(x + x_s). \tag{6.3}$$

With the necessary superpaths defined, the main theorem will now be stated and proved.

Theorem 6.2. For $a = 1, \dots, m$ let ψ^a be the operator

$$\psi^a = \theta^a + \frac{\partial}{\partial \theta^a} \tag{6.4}$$

acting on the space $L^{2\prime}(\mathbb{R}^{m,m}_L)$. Also let ϕ be a C^5 function on $\mathbb{R}^{m,m}$ which is linear in the odd argument and has all derivatives up to fifth order uniformly bounded. Then, if Q acts on $L^{2\prime}(\mathbb{R}^{m,m}_L)$ with

$$Q = \psi^a \left(e^i_a(x)\partial_i\right) + \phi(x, \psi) \tag{6.5}$$

$H = Q^2$, $F \in C^{5\prime}(\mathbb{R}^{m,m}_L)$ and $(x, \theta) \in \mathbb{R}^{m,n}_L$,

$$\exp\left(-Ht - Q\tau\right) F(x,\theta) = \mathbb{E}\left\{\exp\left[\int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, \phi\left(x + z_S + \frac{\mathrm{i}}{\sqrt{2}}\sigma\eta_s, \theta + \zeta_S\right)\right] \right.$$
$$\left. \times F\left(x + z_T + \frac{\mathrm{i}}{\sqrt{2}}\tau\eta_t, \theta + \theta_t\right)\right\}. \tag{6.6}$$

Proof. Let $\overline{U}_{t,\tau}$ be the operator on $L^{2\prime}(\mathbb{R}^{m,m}_L)$ defined by completion of $U_{t,\tau}$: $C_0^{\infty\prime}(\mathbb{R}^{m,m}_L) \to L^{2\prime}(\mathbb{R}^{m,m}_L)$ with

$$U_{t,\tau}\, G(x,\theta) = \mathbb{E}\left\{\exp\left[\int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, \phi\left(x + z_S + \frac{\mathrm{i}}{\sqrt{2}}\sigma\eta_s, \theta + \zeta_S\right)\right] \right.$$
$$\left. \times\, G(x + z_T + \frac{\mathrm{i}}{\sqrt{2}}\tau\eta_t, \theta + \theta_t)\right\} \tag{6.7}$$

where $G \in C_0^{\infty\prime}(\mathbb{R}^{m,m}_L)$. Then, applying the product Ito formula to the integrand in $U_{t,\tau}\, f(x,\theta)$ and taking expectations gives, after some algebra,

$$U_{t,\tau}\, G(x,\theta) - G(x,\theta) = \int_0^\tau \mathrm{d}\sigma \int_0^t \mathrm{d}s\, U_{s,\sigma} QG(x,\theta) - 2\sigma U_{s,\sigma} HG(x,\theta). \tag{6.8}$$

Thus

$$D_T U_{t,\tau} G(x,\theta) = U_{t,\tau}(Q - 2\tau H)G(x,\theta) \tag{6.9}$$

and hence, using equation (4.7) and appealing to the uniqueness of solutions to differential equations such as (6.9), one may deduce that

$$U_{t,\tau} G(x,\theta) = \exp(-Ht - Q\tau)G(x,\theta) \tag{6.10}$$

as required.

This theorem completes the results to be given in the first of these two papers. In the second paper the analysis developed here is applied to various geometric structures such as supermanifolds and spin bundles. In particular, solutions to superspace stochastic differential equations are used to make the path integration step in the supersymmetric proofs of the Atiyah–Singer index theorem [6, 12] rigorous.

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